

EXTENSION THEOREMS FOR SELF-DUAL CODES OVER RINGS AND NEW BINARY SELF-DUAL CODES

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ABSTRACT. In this work, extension theorems are generalized to self-dual codes over rings and as applications many new binary self-dual extremal codes are found from self-dual codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ for $m = 1, 2$. The duality and distance preserving Gray maps from $\mathbb{F}_4 + u\mathbb{F}_4$ to $(\mathbb{F}_2 + u\mathbb{F}_2)^2$ and \mathbb{F}_2^4 are used to obtain self-dual codes whose binary Gray images are $[64, 32, 12]$ -extremal self-dual. An $\mathbb{F}_2 + u\mathbb{F}_2$ -extension is used and as binary images, 178 extremal binary self-dual codes of length 68 with new weight enumerators are obtained. Especially the first examples of codes with $\gamma = 3$ and many codes with the rare $\gamma = 4, 6$ parameters are obtained. In addition to these, two hundred fifty doubly even self dual $[96, 48, 16]$ -codes with new weight enumerators are obtained from four-circulant codes over $\mathbb{F}_4 + u\mathbb{F}_4$. New extremal doubly even binary codes of lengths 80 and 88 are also found by the $\mathbb{F}_2 + u\mathbb{F}_2$ -lifts of binary four circulant codes and a corresponding result about 3-designs is stated.

1. INTRODUCTION

The construction of extremal binary self-dual codes has generated a considerable interest among researchers recently. The connection of these codes to designs, lattices and other such mathematical objects has been a source of motivation for this interest. Several construction methods have been employed for this purpose. Among the most common ones, we can mention double and bordered double-circulant constructions, constructions with a specific automorphism group, and recently ring constructions using different rings of characteristic 2. We refer the reader to [3, 4, 7, 9, 10, 12, 13, 15, 20] and [21] for more on these constructions.

Ling and Sole studied Type II codes over the ring $\mathbb{F}_4 + u\mathbb{F}_4$ in [16], which was later generalized to the ring $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ in [1]. These rings behave similar to the oft-studied ring $\mathbb{F}_2 + u\mathbb{F}_2$ in the literature. The common theme in the aforementioned works is that a distance and duality preserving Gray map can be defined that takes codes over those rings to binary codes, preserving the linearity, the weight distribution and the duality.

Harada and Kim give two different extension methods in [11] and [15] respectively for binary self-dual codes. Both methods describe how a binary self-dual code of length n can be extended to obtain a binary self-dual code of length $n + 2$.

In this work we generalize the extension methods described on the binary field to any binary ring(i.e., a ring of characteristic 2). With this method we extend self-dual codes over binary rings to further lengths which correspond to a more diverse set of lengths. Also with the rich algebraic structure of the ring, we have a

2000 *Mathematics Subject Classification.* Primary:94B05, Secondary:94B99.

Key words and phrases. extremal self-dual codes, Gray maps, four circulant codes, extension theorems.

better chance to get good self-dual codes. The binary rings that we use are mainly $\mathbb{F}_4 + u\mathbb{F}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$ as we already have distance and duality-preserving Gray maps for these rings. Using these methods we were able to obtain 178 new extremal binary self-dual codes of length 68 and 14 new extremal codes of length 80.

The rest of the paper is organized as follows: Preliminaries about codes over $\mathbb{F}_4 + u\mathbb{F}_4$ and the distance and duality-preserving Gray maps are given in section 2. In section 3, we give constructions for binary self-dual codes of length 64 coming from the Gray images of four-circulant self-dual codes over $\mathbb{F}_4 + u\mathbb{F}_4$. In section 4, we describe the ring extension methods to extend self-dual codes over binary rings. In section 5, we apply the ring extension to codes obtained in section 3 to obtain a number of extremal binary self-dual codes of length 68 with new parameters in their weight enumerators. In section 6, we describe constructions of extremal binary self-dual codes of length 80 and 88 as well as new Type II codes of length 96 from codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ for $m = 1, 2$.

2. PRELIMINARIES

Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ be the quadratic field extension of \mathbb{F}_2 , where $\omega^2 + \omega + 1 = 0$. The ring $\mathbb{F}_4 + u\mathbb{F}_4$ defined via $u^2 = 0$ is a commutative binary ring of size 16. We may easily observe that it is isomorphic to $\mathbb{F}_2[\omega, u] / \langle u^2, \omega^2 + \omega + 1 \rangle$. The ring has a unique non-trivial ideal $\langle u \rangle = \{0, u, u\omega, u + u\omega\}$. Note that $\mathbb{F}_4 + u\mathbb{F}_4$ can be viewed as an extension of $\mathbb{F}_2 + u\mathbb{F}_2$ and so we can describe any element of $\mathbb{F}_4 + u\mathbb{F}_4$ in the form $\omega a + \bar{\omega}b$ uniquely, where $a, b \in \mathbb{F}_2 + u\mathbb{F}_2$.

A code C of length n over $\mathbb{F}_4 + u\mathbb{F}_4$ is an $(\mathbb{F}_4 + u\mathbb{F}_4)$ -submodule of $(\mathbb{F}_4 + u\mathbb{F}_4)^n$. Elements of the code C are called codewords of C . Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be two elements of $(\mathbb{F}_4 + u\mathbb{F}_4)^n$. The duality is understood in terms of the Euclidean inner product; $\langle x, y \rangle_E = \sum x_i y_i$. The dual C^\perp of the code C is defined as

$$C^\perp = \{x \in (\mathbb{F}_4 + u\mathbb{F}_4)^n \mid \langle x, y \rangle_E = 0 \text{ for all } y \in C\}.$$

We say that C is self-dual if $C = C^\perp$. Let us recall the following Gray Maps from [8] and [6];

$$\psi_{\mathbb{F}_4} : (\mathbb{F}_4)^n \rightarrow (\mathbb{F}_2)^{2n} \quad \left\| \begin{array}{l} \varphi_{\mathbb{F}_2 + u\mathbb{F}_2} : (\mathbb{F}_2 + u\mathbb{F}_2)^n \rightarrow \mathbb{F}_2^{2n} \\ a\omega + b\bar{\omega} \mapsto (a, b), \quad a, b \in \mathbb{F}_2^n \end{array} \right. \quad \left\| \begin{array}{l} a + bu \mapsto (b, a + b), \quad a, b \in \mathbb{F}_2^n. \end{array} \right.$$

In [16], those were generalized to the following Gray maps;

$$\psi_{\mathbb{F}_4 + u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \rightarrow (\mathbb{F}_2 + u\mathbb{F}_2)^{2n} \quad \left\| \begin{array}{l} \varphi_{\mathbb{F}_4 + u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \rightarrow \mathbb{F}_4^{2n} \\ a\omega + b\bar{\omega} \mapsto (a, b), \quad a, b \in (\mathbb{F}_2 + u\mathbb{F}_2)^n \end{array} \right. \quad \left\| \begin{array}{l} a + bu \mapsto (b, a + b), \quad a, b \in \mathbb{F}_4^n \end{array} \right.$$

Note that these Gray maps preserve orthogonality in the respective alphabets, for the details we refer to [16]. The binary codes $\varphi_{\mathbb{F}_2 + u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$ and $\psi_{\mathbb{F}_4} \circ \varphi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$ are equivalent to each other. The Lee weight of an element in $\mathbb{F}_4 + u\mathbb{F}_4$ is defined to be the Hamming weight of its binary image under any of the previously mentioned compositions of the maps. A self-dual code is said to be of Type II if the Lee weights of all codewords are multiples of 4, otherwise it is said to be of Type I.

Proposition 2.1. ([16]) *Let C be a code over $\mathbb{F}_4 + u\mathbb{F}_4$. If C is self-orthogonal, so are $\psi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$. C is a Type I (resp. Type II) code over $\mathbb{F}_4 + u\mathbb{F}_4$ if and only if $\varphi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$ is a Type I (resp. Type II) \mathbb{F}_4 -code, if and only if $\psi_{\mathbb{F}_4 + u\mathbb{F}_4}(C)$ is a Type I (resp. Type II) $\mathbb{F}_2 + u\mathbb{F}_2$ -code. Furthermore, the*

minimum Lee weight of C is the same as the minimum Lee weight of $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.

Corollary 2.2. *Suppose that C is a self-dual code over $\mathbb{F}_4 + u\mathbb{F}_4$ of length n and minimum Lee distance d . Then $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a binary $[4n, 2n, d]$ self-dual code. Moreover, C and $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ have the same weight enumerator. If C is Type I (Type II), then so is $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.*

An upper bound on the minimum Hamming distance of a binary self-dual code is as follows:

Theorem 2.3. ([18]) *Let $d_I(n)$ and $d_{II}(n)$ be the minimum distance of a Type I and Type II binary code of length n , respectively. Then*

$$d_{II}(n) \leq 4 \lfloor \frac{n}{24} \rfloor + 4$$

and

$$d_I(n) \leq \begin{cases} 4 \lfloor \frac{n}{24} \rfloor + 4 & \text{if } n \not\equiv 22 \pmod{24} \\ 4 \lfloor \frac{n}{24} \rfloor + 6 & \text{if } n \equiv 22 \pmod{24}. \end{cases}$$

Self-dual codes meeting these bounds are called *extremal*. Throughout the text we obtain extremal Type I binary codes of lengths 64 and 68 and extremal Type II codes of lengths 80 and 88. The existence of extremal Type II codes of length 96 is as yet unknown. But we get Type II codes of parameters $[96, 48, 16]$, which is the best known parameter at the moment.

3. $[64, 32, 12]_2$ SINGLY-EVEN CODES AS IMAGES OF $\mathbb{F}_4 + u\mathbb{F}_4$ -LIFTS OF CODES OVER \mathbb{F}_4

The double circulant and bordered double circulant constructions are quite commonly used constructions in the literature for self-dual codes. However there is a variation of these constructions called the four circulant construction which has recently been introduced and used in the context of self-dual codes. We will apply the construction here. The four circulant construction was applied to the ring $\mathbb{F}_2 + u\mathbb{F}_2$ in [13] to obtain extremal binary self-dual codes. The main theorem that can exactly be extended to include the ring $\mathbb{F}_4 + u\mathbb{F}_4$ is the following:

Theorem 3.1. ([13], with \mathbb{F}_2 replaced by \mathbb{F}_4) *Let C be the linear code over $\mathbb{F}_4 + u\mathbb{F}_4$ of length $4n$ generated by the four circulant matrix*

$$G := \left[I_{2n} \mid \begin{array}{cc} A & B \\ B^T & A^T \end{array} \right]$$

where A and B are circulant $n \times n$ matrices over $\mathbb{F}_4 + u\mathbb{F}_4$ satisfying $AA^T + BB^T = I_n$. Then C is self-dual.

The proof being exactly the same as the case of $\mathbb{F}_2 + u\mathbb{F}_2$, is omitted here.

Now, our aim is to find extremal binary self-dual codes using the four circulant construction over $\mathbb{F}_4 + u\mathbb{F}_4$. This requires a restriction on the minimum weight. To reduce the search field we will consider the projection $\mu : \mathbb{F}_4 + u\mathbb{F}_4 \rightarrow \mathbb{F}_4$ by letting $\mu(a + bu) = a$ for all $a, b \in \mathbb{F}_4$. This map then can be extended in a natural way to $(\mathbb{F}_4 + u\mathbb{F}_4)^n$. It can easily be shown that μ preserves duality and because of the type of the matrix, we can say that if C is a four circulant self-dual code generated by a matrix G of the form given in Theorem 3.1, then $\mu(C)$ will also be a four circulant self dual code over \mathbb{F}_4 generated by the matrix $\mu(G)$. Thus any

TABLE 1. Four circulant codes over \mathbb{F}_4

| \mathcal{C}_i | r_A | r_B | $\psi_{\mathbb{F}_4}(C)$ | $ Aut(C) $ |
|-----------------|--|--|--------------------------|--------------------|
| \mathcal{C}_1 | $(1, \omega, \omega, 0)$ | $(\omega, 1 + \omega, 1 + \omega, \omega)$ | $[32, 16, 8]_2$ | $2^{12}3 \times 7$ |
| \mathcal{C}_2 | $(1, 0, 1, \omega)$ | $(0, 0, 1 + \omega, 0)$ | $[32, 16, 6]_2$ | $2^9 3^2 5$ |
| \mathcal{C}_3 | $(\omega, \omega, 1 + \omega, 1 + \omega)$ | $(1 + \omega, \omega, 0, 0)$ | $[32, 16, 6]_2$ | $2^9 3^2 5$ |
| \mathcal{C}_4 | $(1, 0, 1, 1 + \omega)$ | $(0, 0, \omega, 0)$ | $[32, 16, 6]_2$ | $2^9 3^2 5$ |
| \mathcal{C}_5 | $(1 + \omega, 0, 1 + \omega, \omega)$ | $(0, 0, 1 + \omega, 0)$ | $[32, 16, 6]_2$ | $2^9 3^2 5$ |

four circulant self-dual code over $\mathbb{F}_4 + u\mathbb{F}_4$ can be viewed as a lift of a four circulant self-dual code over \mathbb{F}_4 of the same length. The following theorem, an analogue of which can also be found in [13] reduces the search field quite considerably:

Theorem 3.2. ([13], with \mathbb{F}_2 replaced by \mathbb{F}_4) Suppose C is a linear code over $\mathbb{F}_4 + u\mathbb{F}_4$ and that $C' = \mu(C)$ is its projection to \mathbb{F}_4 . With d and d' representing the minimum Lee distances of C and C' respectively, we have $d \leq 2d'$.

So, to construct binary extremal self-dual codes of length 64, we need self-dual codes over $\mathbb{F}_4 + u\mathbb{F}_4$ of length 16 and minimum Lee weight 12. However the projections of four circulant self-dual codes over $\mathbb{F}_4 + u\mathbb{F}_4$ being four circulant self-dual codes over \mathbb{F}_4 , by Theorem 3.2 we need four circulant self-dual codes over \mathbb{F}_4 of minimum Lee weight at least 6. A complete classification of all four-circulant self-dual codes over \mathbb{F}_4 of length 16 can be done by considering all possible first rows for the matrices A and B , denoted henceforth by r_A and r_B , which requires a search over 4^8 possible matrices, only a portion of which will be self-dual with minimum Lee weight ≥ 6 . Lifting these to $\mathbb{F}_4 + u\mathbb{F}_4$, we see that only the codes listed in Table 1 have resulted in self-dual codes with extremal binary images. There are two possibilities for the weight enumerators of extremal singly-even $[64, 32, 12]_2$ codes ([2]):

$$\begin{aligned} W_{64,1} &= 1 + (1312 + 16\beta)y^{12} + (22016 - 64\beta)y^{14} + \dots, 14 \leq \beta \leq 284, \\ W_{64,2} &= 1 + (1312 + 16\beta)y^{12} + (23040 - 64\beta)y^{14} + \dots, 0 \leq \beta \leq 277. \end{aligned}$$

The theoretical values for β have not all been constructed yet. Most recently, codes with $\beta = 25, 39, 53$ and 60 in $W_{64,1}$ and $\beta = 51$ and 58 in $W_{64,2}$ are constructed in [19], a code with $\beta = 80$ in $W_{64,2}$ is constructed in [13]. Together with these, codes exist with weight enumerators $\beta = 14, 18, 22, 25, 32, 36, 39, 44, 46, 53, 60$ and 64 in $W_{64,1}$ and for $\beta = 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23, 24, 25, 28, 19, 30, 32, 33, 36, 37, 38, 40, 41, 44, 48, 51, 52, 56, 58, 64, 72, 80, 88, 96, 104, 108, 112, 114, 118, 120$ and 184 in $W_{64,2}$.

In order to fit the upcoming tables regarding the results, we label the elements of $\mathbb{F}_4 + u\mathbb{F}_4$ as follows;

| | | | | | | | |
|-------|---------------|-------|-------------------|-------|------------------------|-------|----------------------------|
| z_1 | 0 | a_1 | 1 | b_1 | ω | c_1 | $1 + \omega$ |
| z_2 | u | a_2 | $1 + u$ | b_2 | $\omega + u$ | c_2 | $1 + \omega + u$ |
| z_3 | $u\omega$ | a_3 | $1 + u\omega$ | b_3 | $\omega + u\omega$ | c_3 | $1 + \omega + u\omega$ |
| z_4 | $u + u\omega$ | a_4 | $1 + u + u\omega$ | b_4 | $\omega + u + u\omega$ | c_4 | $1 + \omega + u + u\omega$ |

We lift the \mathbb{F}_4 -codes given in Table 1 to $\mathbb{F}_4 + u\mathbb{F}_4$, as a result of which we obtain extremal binary self-dual codes of length 64 as given in Table 2.

TABLE 2. The $\mathbb{F}_4 + u\mathbb{F}_4$ -lifts of \mathcal{C}_i and the β values of the binary images

| code | | first row of A | first row of B | β in $W_{64,2}$ | $ Aut(C) $ |
|-----------------|-----------------|------------------------|------------------------|-----------------------|------------|
| \mathcal{J}_1 | \mathcal{C}_1 | (a_2, b_3, b_1, z_4) | (b_4, c_4, c_1, b_2) | 48 | 2^5 |
| \mathcal{J}_2 | \mathcal{C}_1 | (a_3, b_2, b_4, z_1) | (b_3, c_3, c_2, b_1) | 52 | 2^5 |
| \mathcal{K}_1 | \mathcal{C}_2 | (a_4, z_3, a_3, b_1) | (z_2, z_4, c_3, z_1) | 8 | 2^5 |
| \mathcal{K}_2 | \mathcal{C}_2 | (a_1, z_4, a_2, b_4) | (z_2, z_4, c_4, z_1) | 40 | 2^5 |
| \mathcal{K}_3 | \mathcal{C}_2 | (a_2, z_4, a_1, b_4) | (z_2, z_4, c_1, z_1) | 40 | 2^5 |
| \mathcal{K}_4 | \mathcal{C}_2 | (a_2, z_2, a_1, b_2) | (z_4, z_3, c_1, z_2) | 40 | 2^5 |
| \mathcal{K}_5 | \mathcal{C}_2 | (a_2, z_4, a_1, b_2) | (z_2, z_1, c_1, z_4) | 40 | 2^5 |
| \mathcal{L}_1 | \mathcal{C}_3 | (b_2, b_4, c_2, c_3) | (c_1, b_1, z_2, z_2) | 0 | 2^6 |
| \mathcal{L}_2 | \mathcal{C}_3 | (b_1, b_4, c_1, c_3) | (c_2, b_2, z_1, z_1) | 0 | 2^5 |
| \mathcal{L}_3 | \mathcal{C}_3 | (b_1, b_3, c_1, c_4) | (c_1, b_2, z_2, z_1) | 4 | 2^5 |
| \mathcal{L}_4 | \mathcal{C}_3 | (b_4, b_2, c_2, c_1) | (c_2, b_1, z_1, z_3) | 8 | 2^4 |
| \mathcal{L}_5 | \mathcal{C}_3 | (b_1, b_4, c_2, c_3) | (c_1, b_3, z_3, z_4) | 12 | 2^4 |
| \mathcal{L}_6 | \mathcal{C}_3 | (b_2, b_4, c_3, c_2) | (c_3, b_1, z_1, z_4) | 24 | 2^5 |
| \mathcal{L}_7 | \mathcal{C}_3 | (b_1, b_4, c_1, c_2) | (c_3, b_1, z_2, z_2) | 28 | 2^4 |
| \mathcal{L}_8 | \mathcal{C}_3 | (b_2, b_2, c_1, c_1) | (c_3, b_1, z_1, z_2) | 36 | 2^5 |
| \mathcal{M}_1 | \mathcal{C}_4 | (a_1, z_4, a_2, c_1) | (z_1, z_4, b_2, z_2) | 40 | 2^5 |
| \mathcal{M}_2 | \mathcal{C}_4 | (a_1, z_4, a_2, c_1) | (z_1, z_4, b_4, z_2) | 40 | 2^5 |
| \mathcal{M}_3 | \mathcal{C}_4 | (a_1, z_4, a_2, c_1) | (z_1, z_4, b_2, z_2) | 40 | 2^5 |
| \mathcal{N}_1 | \mathcal{C}_5 | (c_1, z_1, c_4, b_1) | (z_1, z_2, c_2, z_4) | 32 | 2^5 |

4. EXTENSION METHODS FOR SELF-DUAL CODES OVER BINARY RINGS

In the sequel, let S be a commutative ring of characteristic 2 with identity.

Theorem 4.1. *Let C be a self-dual code over S of length n and $G = (r_i)$ be a $k \times n$ generator matrix for C , where r_i is the i -th row of G , $1 \leq i \leq k$. Let c be a unit in S such that $c^2 = 1$ and X be a vector in S^n with $\langle X, X \rangle = 1$. Let $y_i = \langle r_i, X \rangle$ for $1 \leq i \leq k$. Then the following matrix*

$$\left[\begin{array}{cc|c} 1 & 0 & X \\ \hline y_1 & cy_1 & r_1 \\ \vdots & \vdots & \vdots \\ y_k & cy_k & r_k \end{array} \right],$$

generates a self-dual code D over S of length $n + 2$.

A more specific extension method which can easily be applied to circulant codes may be given as follows:

Theorem 4.2. *Let C be a self-dual code generated by $G = [I_n | A]$ over S . If the sum of the elements in i -th row of A is r_i then the matrix:*

$$G^* = \left[\begin{array}{cc|cccc} 1 & 0 & x_1 & \dots & x_n & 1 & \dots & 1 \\ \hline y_1 & cy_1 & & & & & & \\ \vdots & \vdots & & & & & & \\ y_n & cy_n & & & & & & \end{array} \right],$$

where $y_i = x_i + r_i$, c is a unit with $c^2 = 1$, $\langle X, X \rangle = 1 + n$ and $X = (x_1, \dots, x_n)$, generates a self-dual code C^* over S .

Remark 4.3. As can be seen, these extension theorems generalize the binary extension theorems given in [11] and [15]. The proofs being exactly analogous, have been omitted here.

5. NEW EXTREMAL BINARY SELF DUAL CODES OF LENGTH 68 FROM $\mathbb{F}_2 + u\mathbb{F}_2$ EXTENSIONS

The weight enumerator of a self-dual $[68, 34, 12]_2$ code is in one of the following forms ([5]):

$$\begin{aligned} W_{68,1} &= 1 + (442 + 4\beta)y^{12} + (10864 - 8\beta)y^{14} + \dots, \\ W_{68,2} &= 1 + (442 + 4\beta)y^{12} + (14960 - 8\beta - 256\gamma)y^{14} + \dots \end{aligned}$$

where β and γ are parameters. Tsai et al. constructed a substantial number of codes in both possible weight enumerators in [21]. Recently, 32 new codes are obtained in [14] and 28 new codes including the first examples with $\gamma = 4$ and $\gamma = 6$ in $W_{68,2}$ are obtained in [12]. Together with the ones in [12, 14] codes exists for $W_{68,2}$ when $\gamma = 0$ and $\beta = 38, 40, 44, 45, 47, \dots, 136, 138, 139, 170, 204, 238, 272$; $\gamma = 1$ and $\beta = 61, 63, 64, 65, 72, 73, 76, 77, 79, 81, \dots, 115, 118, 126, 129, 132, 133, 138, 140, 142, 146$; $\gamma = 2$ and $\beta = 65, 71, 77, 82, 84, 86, 88, 93, 94, 96, 99, 109, 123, 130, 132, 134, 140, 142, 146, 152$ or $\beta \in \{2m \mid 51 \leq m \leq 63\}$; $\gamma = 4$ and $\beta = 116, 122, 124, 128, 140, 142, 152$ and $\gamma = 6$ with $\beta = 176$. The extension methods in section 4 are applied to the $\psi_{\mathbb{F}_4 + u\mathbb{F}_4}$ -images of the codes in table 2. Throughout the tables 3-7, 9 the codes are generated over $\mathbb{F}_2 + u\mathbb{F}_2$ by the matrices of the following form;

$$\left[\begin{array}{cc|c} 1 & 0 & X \\ \hline y_1 & cy_1 & \\ \vdots & \vdots & \psi_{\mathbb{F}_4 + u\mathbb{F}_4}(C_i) \\ y_k & cy_k & \end{array} \right].$$

The second extension theorem is used to obtain the results tabulated in Table 8. As binary images of all these codes we were able to obtain 181 new codes in $W_{68,2}$, which are listed in the tables 3-9. More precisely, 14 codes with $\gamma = 0$ in tables 3,8 and 9, 47 codes with $\gamma = 1$ listed in tables 4,8 and 9, 42 codes with $\gamma = 2$ in Table 5, 37 codes with $\gamma = 3$ in tables 6,8 and 9 21 codes with $\gamma = 4$ listed in Table 7 and 5 codes with $\gamma = 6$ which are listed in Table 3. In order to save space $1 + u$ in X are replaced by 3 in tables. Note that the codes with $\gamma = 3$ in their weight enumerators are the first examples in the literature of that parameter.

5.1. New codes from a previously constructed code. Karadeniz et al. constructed four circulant codes of length 32 over $\mathbb{F}_2 + u\mathbb{F}_2$ whose Gray images are extremal singly-even binary codes of length 64 in [13]. One of these codes has a new weight enumerator in $W_{64,2}$ with $\beta = 80$. Since the β -value of this code is greater than that of the codes we were able to construct, we apply the extension methods to this code. We were able to obtain a substantial number of binary extremal codes of length 68 with new weight enumerators in $W_{68,2}$ as Gray images of $\mathbb{F}_2 + u\mathbb{F}_2$ -extensions.

Let C_{64} be the four circulant code over $\mathbb{F}_2 + u\mathbb{F}_2$ with $r_A = (u, 0, 0, 0, u, 1, u, 1 + u)$ and $r_B = (u, u, 0, 1, 1, 1 + u, 1 + u, 1 + u)$. The extension method in Theorem 4.2

TABLE 3. $[68, 34, 12]$ codes with $\gamma = 0$ and $\gamma = 6$ in $W_{68,2}$ (16 codes)

| Code | X | c | γ | β |
|-----------------|--------------------------------------|-------|----------|---------|
| \mathcal{L}_3 | $[111013303300031u31u3uu10uuu000u1]$ | $1+u$ | 0 | 46 |
| \mathcal{M}_1 | $[u13u3u33110uu11u10110000uu13u100]$ | 1 | 0 | 137 |
| \mathcal{M}_1 | $[3uu031uu30u13u3u0u31u33111113013]$ | $1+u$ | 0 | 141 |
| \mathcal{K}_3 | $[u1030u3u1u1u033301uu333101u30101]$ | 1 | 0 | 142 |
| \mathcal{J}_2 | $[0u1uuuu11u331013u1u130u113u131uu]$ | 1 | 0 | 143 |
| \mathcal{M}_1 | $[1300uu030u03310031130u30000u303u]$ | 1 | 0 | 145 |
| \mathcal{J}_2 | $[u11u000u31uu00030u33100303100103]$ | $1+u$ | 0 | 147 |
| \mathcal{N}_1 | $[303010u310u010u0u01031131001033u]$ | 1 | 0 | 148 |
| \mathcal{J}_2 | $[1001uu1103111u3031013113001130u0]$ | 1 | 0 | 149 |
| \mathcal{J}_2 | $[3u1u033u10310330011u031003010130]$ | 1 | 0 | 151 |
| \mathcal{J}_1 | $[03013310u03u313330011u0u13113030]$ | 1 | 0 | 153 |
| \mathcal{L}_6 | $[101110u1301303u033311033u033uu30]$ | $1+u$ | 6 | 138 |
| \mathcal{L}_6 | $[uu1u1u11301313uu03331101303u3uu1]$ | 1 | 6 | 154 |
| \mathcal{L}_6 | $[30111uuuu0033010330003301301010u]$ | 1 | 6 | 156 |
| \mathcal{L}_6 | $[30u30133113uuuu3u3u0u111u3300101]$ | 1 | 6 | 158 |
| \mathcal{L}_6 | $[uu311333001u033u010110011000u131]$ | 1 | 6 | 162 |

is applied to \mathcal{C}_{64} and 24 new codes in $W_{68,2}$ are obtained as Gray images of the extensions, so the codes in Table 8 are the Gray images of the codes generated by;

$$\left[\begin{array}{cc|cccccc} 1 & 0 & x_1 & \dots & x_{16} & 1 & \dots & 1 \\ y_1 & cy_1 & & & & & & \\ \vdots & \vdots & & & & & & \\ y_{16} & cy_{16} & & I_{16} & & & M & \end{array} \right]$$

where M is the four-circulant matrix corresponding to \mathcal{C}_{64} and $X = (x_1, x_2, \dots, x_{16})$ is a random vector over $\mathbb{F}_2 + u\mathbb{F}_2$ which satisfies $\langle X, X \rangle = 1$ and $y_i = x_i + 1 + u$.

In addition to these, by applying the extension method in Theorem 4.1 to \mathcal{C}_{64} we were able obtain 13 new codes which are listed in Table 9.

6. NEW DOUBLY EVEN BINARY CODES OF LENGTHS 80, 88 AND 96

In this section, specific four circulant codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ are considered. The codes are constructed as lifts of binary codes. Binary images of the codes are extremal doubly-even codes of length 80 and 88, the inequivalence of the codes is verified by the invariants. As a result, we obtain 14 new extremal self-dual $[80, 40, 16]_2$ Type II codes and first extremal Type II codes of length 88 with an automorphism group of order 44. In addition, Type II $[96, 48, 16]_2$ codes with new weight enumerators are obtained by applying the method to $\mathbb{F}_4 + u\mathbb{F}_4$.

6.1. New extremal doubly even $[80, 40, 16]_2$ codes. The weight enumerator of a doubly-even $[80, 40, 16]_2$ code is uniquely determined as $1 + 97565y^{16} + 12882688y^{20} + \dots$ [5]. The extended quadratic residue code QR_{80} is the first doubly-even $[80, 40, 16]_2$ code [17]. In [4], Dontcheva and Harada constructed 11 new codes with an automorphism of order 19. Later Gulliver and Harada constructed 10 new codes by double circulant construction in [10]. We construct 14 new codes as Gray images

TABLE 4. $[68, 34, 12]$ codes with $\gamma = 1$ in $W_{68,2}$ (30 codes)

| Code | X | c | β |
|-----------------|--------------------------------------|---------|---------|
| \mathcal{L}_2 | $[13u30uuu3u10uuu33111311uuu010uu1]$ | 1 | 54 |
| \mathcal{L}_2 | $[011u031u113311uu13u310130u033u01]$ | 1 | 56 |
| \mathcal{L}_1 | $[1130u311u33uu31u3uu01103311uu031]$ | 1 | 58 |
| \mathcal{L}_1 | $[310133uuu1uu310u330uuu0101000u1u]$ | $1 + u$ | 60 |
| \mathcal{L}_2 | $[10u0033u03331131000uu033u11133uu]$ | $1 + u$ | 62 |
| \mathcal{L}_1 | $[333u111131uuu0u011uu0uu1uu1uu301]$ | 1 | 66 |
| \mathcal{L}_1 | $[u3u003u1u11u1uu1u101310u00003001]$ | $1 + u$ | 68 |
| \mathcal{L}_1 | $[31100103u1u10313u0u01u1u3101u033]$ | 1 | 70 |
| \mathcal{L}_1 | $[303010033030110uuu003uu10313uu11]$ | $1 + u$ | 74 |
| \mathcal{L}_3 | $[0u1u313u1u30033031u3311u10u30u01]$ | 1 | 75 |
| \mathcal{L}_1 | $[113u330u133010u3111uu1110103uu10]$ | 1 | 78 |
| \mathcal{L}_1 | $[030u011331000000113uu13303003131]$ | $1 + u$ | 80 |
| \mathcal{N}_1 | $[3101u3311133uu001001313u11311uu3]$ | 1 | 119 |
| \mathcal{M}_1 | $[0011u3uu013u3000101u01130u1101u1]$ | 1 | 120 |
| \mathcal{N}_1 | $[0u33103001u31u101310333011101111]$ | 1 | 123 |
| \mathcal{N}_1 | $[131u1u03u111u310u0u10333u3uuu033]$ | 1 | 125 |
| \mathcal{N}_1 | $[u303uu3313u3uu0003331011uu1u0313]$ | $1 + u$ | 135 |
| \mathcal{N}_1 | $[30u0uuu0u11330130u1311303uu1003u]$ | $1 + u$ | 137 |
| \mathcal{K}_2 | $[33u00313uu1u1uuu11uu3u00u100u11u]$ | $1 + u$ | 139 |
| \mathcal{M}_1 | $[133u030u300uu031u03u1333u0111031]$ | 1 | 141 |
| \mathcal{K}_2 | $[uu3u030u03uu01u1u01110u111u33000]$ | $1 + u$ | 143 |
| \mathcal{J}_2 | $[113300311310010013300u1000uu0100]$ | $1 + u$ | 144 |
| \mathcal{K}_2 | $[01103u3uu3110uu003u1u1u3300uu111]$ | $1 + u$ | 145 |
| \mathcal{J}_2 | $[uuu33uu303100101111u10uu03u33u3u]$ | 1 | 147 |
| \mathcal{M}_1 | $[31u013u1130031111uu00uu1001u1013]$ | 1 | 149 |
| \mathcal{M}_1 | $[3uu3u3u03uu0uu03330u1uu11u30u033]$ | $1 + u$ | 150 |
| \mathcal{J}_2 | $[1u03u0u33133333133u3u11u1001111u]$ | 1 | 151 |
| \mathcal{K}_2 | $[u1u13u0311033uu11010u1031u013u30]$ | $1 + u$ | 153 |
| \mathcal{J}_2 | $[u01u33u1133u03uu13311u31u1u3uu11]$ | 1 | 155 |
| \mathcal{J}_1 | $[3u3u1331u30113313133u300110u1131]$ | $1 + u$ | 159 |

of four circulant $\mathbb{F}_2 + u\mathbb{F}_2$ -lifts of $[40, 20, 8]_2$ -codes given in Table 10. The codes which have an automorphism group of orders 40 and 240 are the first such codes. The inequivalence of the codes is checked by the invariants. Let $c_1, c_2, \dots, c_{97565}$ be the codewords of weight 16 in an extremal doubly-even $[80, 40, 16]_2$ code. Let $I_j = |\{(c_k, c_l) \mid d(c_k, c_l) = j, k < l\}|$ where d is the Hamming distance. Two codes are inequivalent if their I_{16} -values are different since I_{16} is invariant under a permutation of the coordinates.

As lifts of the codes in Table 10 we obtain new extremal doubly-even codes of length 80 which are listed in Table 11. The codes $\mathcal{L}_{80,6}$, $\mathcal{L}_{80,7}$ and $\mathcal{L}_{80,8}$ in Table 11 have an automorphism group of order 80 and these are inequivalent to such codes $P_{80,2}$, $P_{80,3}$, $P_{80,4}$ and $P_{80,5}$ in [10] since their I_{16} values are 20290440, 20187210, 20201130 and 20034000 respectively. Hence, we have the following theorem:

TABLE 5. $[68, 34, 12]$ codes with $\gamma = 2$ in $W_{68,2}$ (42 codes)

| Code | X | c | β |
|-----------------|--------------------------------------|---------|---------|
| \mathcal{L}_1 | $[0u13u10uu131u311010001011uu30331]$ | $1 + u$ | 68 |
| \mathcal{L}_2 | $[000u3013303uuu11131u3u10uu013u30]$ | $1 + u$ | 74 |
| \mathcal{L}_1 | $[1303301100013333uu303131u11uu300]$ | $1 + u$ | 76 |
| \mathcal{L}_1 | $[u11033001u1u01u13u1u03001u1u11u0]$ | $1 + u$ | 78 |
| \mathcal{L}_1 | $[0011u003u013uu0031303uu3100u0130]$ | $1 + u$ | 80 |
| \mathcal{L}_3 | $[1uu1313u01u3u1uu111u11u31u110uuu]$ | 1 | 85 |
| \mathcal{L}_3 | $[30u30303u33111303u10301u3000103u]$ | 1 | 87 |
| \mathcal{L}_4 | $[0u1uu003010u3u311u31uuu10u0uu003]$ | 1 | 89 |
| \mathcal{L}_1 | $[0030113103u331000uu0133u130033u3]$ | 1 | 90 |
| \mathcal{L}_4 | $[0u0uuu0113303u3u1uu110u1u313u1u1]$ | 1 | 91 |
| \mathcal{L}_1 | $[u0113u11uuu33030313u11u1011uuu1u]$ | $1 + u$ | 92 |
| \mathcal{L}_3 | $[3110u0000113u0310133uu0u3u031u03]$ | $1 + u$ | 95 |
| \mathcal{L}_3 | $[010uu3u313300u31uuu031100u01131u]$ | 1 | 97 |
| \mathcal{L}_1 | $[030010113001u030u11u10u310300u31]$ | 1 | 98 |
| \mathcal{L}_1 | $[0011uu31333u013033u1310011011u01]$ | $1 + u$ | 100 |
| \mathcal{L}_7 | $[100u13u03uuu3303131uu033311u3313]$ | $1 + u$ | 101 |
| \mathcal{L}_4 | $[301u331013100330003u0131030330uu]$ | 1 | 103 |
| \mathcal{L}_3 | $[101u13u310u133u000u0u1u133331u01]$ | $1 + u$ | 105 |
| \mathcal{N}_1 | $[01u10u3333013u3u01030u0uuuu33uuu]$ | 1 | 111 |
| \mathcal{N}_1 | $[33u3u0u11uu3uu0u00u00u1u01u331uu]$ | $1 + u$ | 115 |
| \mathcal{L}_4 | $[3101u103u0100uu1u001133u13011130]$ | 1 | 117 |
| \mathcal{N}_1 | $[u130u031uu10u3101u0031131u0u1001]$ | 1 | 119 |
| \mathcal{L}_4 | $[11313u1111131131u0uu3u0033u03uu0]$ | $1 + u$ | 121 |
| \mathcal{N}_1 | $[011131u0u0u0300001333u33u10u3uu3]$ | $1 + u$ | 125 |
| \mathcal{M}_1 | $[u0uu11u1010u1u1u33010u3uu0u00131]$ | $1 + u$ | 127 |
| \mathcal{M}_1 | $[01u013u3311130011030u30031uu01u3]$ | 1 | 128 |
| \mathcal{M}_1 | $[000uuu0u331010010u301u0u0101000u]$ | $1 + u$ | 129 |
| \mathcal{K}_4 | $[0u00u00333013u30001010u0u1110011]$ | 1 | 131 |
| \mathcal{M}_1 | $[u03u31u31u0u03u0u103u31111uuuu13]$ | $1 + u$ | 133 |
| \mathcal{M}_1 | $[u1030133u033311113u03u0101uu3130]$ | $1 + u$ | 135 |
| \mathcal{K}_3 | $[13001uu31u1310u1u31u0031u101u031]$ | 1 | 136 |
| \mathcal{K}_3 | $[113uu300u0331u0u0u3101u130u1u103]$ | $1 + u$ | 137 |
| \mathcal{K}_4 | $[030u1130u10u0111uu30u1u000133011]$ | 1 | 139 |
| \mathcal{N}_1 | $[u01u0u0u1uuu33131u13u3u013033311]$ | 1 | 144 |
| \mathcal{J}_2 | $[1303111111u33u01301u30031u0000uu]$ | 1 | 145 |
| \mathcal{N}_1 | $[0u311301u0103u3103u3013uuuuu101u]$ | $1 + u$ | 148 |
| \mathcal{K}_3 | $[3133u1u3uu01333u0303uu3u30uu10u0]$ | $1 + u$ | 150 |
| \mathcal{M}_1 | $[03010uuu0u3131uu03uu0u0033011130]$ | 1 | 151 |
| \mathcal{K}_4 | $[0u331100130u111330303u3033u3101u]$ | 1 | 153 |
| \mathcal{K}_3 | $[0uuu03u303130303uu0301uu33uuu0]$ | 1 | 155 |
| \mathcal{K}_3 | $[u30u1u301330030103u0u1003u1u1103]$ | 1 | 158 |
| \mathcal{L}_8 | $[33uu113uu00031u30u3333u031001uu0]$ | 1 | 160 |
| \mathcal{M}_1 | $[3303133u1u1u30uu111003uu010u1uuu]$ | 1 | 162 |

TABLE 6. $[68, 34, 12]$ codes with $\gamma = 3$ in $W_{68,2}$ (34 codes)

| Code | X | c | β |
|-----------------|--------------------------------------|---------|---------|
| \mathcal{L}_2 | $[11101uu0uu113uu001u001u3u0311301]$ | 1 | 88 |
| \mathcal{L}_2 | $[3033311u3uuu31uu301u3u1310u00013]$ | $1 + u$ | 90 |
| \mathcal{L}_2 | $[u330u001013u33u3u333u3101303010u]$ | 1 | 96 |
| \mathcal{L}_3 | $[301113uu10110u3u011uuu00333uuuu1]$ | 1 | 100 |
| \mathcal{L}_2 | $[33u03331u3u1u010031u3333uu3111uu]$ | $1 + u$ | 102 |
| \mathcal{L}_2 | $[30u1u13u000110uuu3u3u113010u1301]$ | 1 | 104 |
| \mathcal{L}_1 | $[3u313u3u3u00133u010013100u011u33]$ | $1 + u$ | 108 |
| \mathcal{K}_1 | $[1u11030uu3303111u3uu03u3100u0030]$ | 1 | 112 |
| \mathcal{L}_3 | $[u1u00uu33u33330uu01uuu133013u1u1]$ | $1 + u$ | 114 |
| \mathcal{L}_3 | $[3u03u01010003u0u0u1303uu0u331000]$ | 1 | 116 |
| \mathcal{L}_4 | $[u03u330uu331303uu0301u0311u3333u]$ | $1 + u$ | 117 |
| \mathcal{L}_5 | $[031u0030u030u013u1u311u111303u33]$ | $1 + u$ | 126 |
| \mathcal{M}_1 | $[31103001113313101uuu1uu13031u10u]$ | $1 + u$ | 127 |
| \mathcal{L}_6 | $[1uu10uuu30133010113uu33303011113]$ | 1 | 128 |
| \mathcal{L}_5 | $[303u00uuu13033uu113u3313011u1uu1]$ | 1 | 130 |
| \mathcal{N}_1 | $[u033301u1u311313133uu31133010030]$ | $1 + u$ | 133 |
| \mathcal{L}_7 | $[31u313u0u0u131u31300u3u3u0u3uuu3]$ | 1 | 136 |
| \mathcal{M}_3 | $[0u100031u010uu331111u0u0u100u000]$ | $1 + u$ | 137 |
| \mathcal{L}_7 | $[33331u033u1u03u0110u1u1uu3u03u33]$ | 1 | 138 |
| \mathcal{L}_7 | $[3uu1333110130uuu01uu0u113310110u]$ | 1 | 140 |
| \mathcal{M}_1 | $[33u11100330u133001u031u00301u110]$ | 1 | 141 |
| \mathcal{L}_6 | $[u1u301u30u03u1u00103011310313u00]$ | $1 + u$ | 142 |
| \mathcal{J}_2 | $[u0030u11100uu30uu1u13u00uu311303]$ | 1 | 144 |
| \mathcal{M}_1 | $[33u11u3103333uu330031u00310uu3uu]$ | $1 + u$ | 145 |
| \mathcal{M}_1 | $[00300uuu111311uu0300100uu001uu1u]$ | $1 + u$ | 147 |
| \mathcal{K}_5 | $[03000013u0u133u030u0uu3131131300]$ | 1 | 148 |
| \mathcal{M}_2 | $[01331u113u0u3331000uu3u11103u3u0]$ | 1 | 149 |
| \mathcal{K}_4 | $[3uu300u0uu0310u031131u01010u11u0]$ | 1 | 153 |
| \mathcal{K}_5 | $[u3u0u133001uu13311u01u100111111]$ | 1 | 154 |
| \mathcal{K}_2 | $[1uu13u3011300u3u3110u03u03311u10]$ | $1 + u$ | 158 |
| \mathcal{M}_2 | $[1uuu33001u03303033uu10u3101u00uu]$ | $1 + u$ | 159 |
| \mathcal{K}_3 | $[103u3010u11uu1u133111033u0u13310]$ | 1 | 160 |
| \mathcal{J}_1 | $[1001133uu3013u1010031u311u30uuu3]$ | 1 | 162 |
| \mathcal{J}_2 | $[u0uu1u10u00330u00u0uu0u100u33330]$ | $1 + u$ | 193 |

Theorem 6.1. *There exist at least 36 extremal doubly-even self-dual codes of length 80.*

By the Assmus-Matson theorem the codewords of weight 16 in an extremal doubly-even code of length 80 form a 3-design.

Theorem 6.2. *There are at least 36 non-isomorphic $3 - (80, 16, 665)$ designs.*

6.2. New extremal doubly even $[88, 44, 16]_2$ codes. There are four circulant $[44, 22, 8]_2$ self-dual codes. We apply the lifting method to one of them and obtain 100 inequivalent extremal doubly even $[88, 44, 16]_2$ codes as the Gray images of

TABLE 7. $[68, 34, 12]$ codes with $\gamma = 4$ in $W_{68,2}$ (21 codes)

| Code | X | c | β |
|-----------------|--------------------------------------|---------|---------|
| \mathcal{L}_2 | $[11031001u00uu0u13u01u10u1u333u31]$ | 1 | 102 |
| \mathcal{L}_1 | $[11u11111u0u3uuuuu110111310u03010]$ | 1 | 110 |
| \mathcal{L}_2 | $[301u131uu3u1133311303u1uu13uu3uu]$ | $1 + u$ | 120 |
| \mathcal{L}_1 | $[1u100313uu3001311u0u01u30131uu33]$ | $1 + u$ | 130 |
| \mathcal{L}_5 | $[u110uu00u3u01113103u11u00u3030u0]$ | 1 | 134 |
| \mathcal{L}_5 | $[u10003uu100u03031u00013333u0u1u1]$ | $1 + u$ | 136 |
| \mathcal{L}_6 | $[u310u000uuu0uu3u101u33111u33003u]$ | $1 + u$ | 138 |
| \mathcal{L}_6 | $[u131u110u130u0u013101u00u1100uu0]$ | $1 + u$ | 150 |
| \mathcal{K}_2 | $[30uu33000u013u30303u13u303033u03]$ | 1 | 154 |
| \mathcal{K}_3 | $[3u33u0u30uu13u003100130311u10u30]$ | $1 + u$ | 156 |
| \mathcal{K}_2 | $[101u010u300u303u3uuuu1u11113u130]$ | 1 | 158 |
| \mathcal{K}_3 | $[10310uu1133u31030331u030010u01u3]$ | 1 | 160 |
| \mathcal{K}_2 | $[131110013uu13uu300u300001313u013]$ | $1 + u$ | 162 |
| \mathcal{K}_2 | $[0u311301u01131u103u30111003u0311]$ | $1 + u$ | 164 |
| \mathcal{K}_5 | $[3301u3133u01u33uu3013u3u31u01u13]$ | $1 + u$ | 166 |
| \mathcal{K}_3 | $[uu0u0u301u1111313u1uuu3u11u110u0]$ | 1 | 168 |
| \mathcal{K}_3 | $[11u0133133u0uu3313u1u0uu0u33330u]$ | 1 | 170 |
| \mathcal{K}_5 | $[1130uu10003313113u1uu1uu300uuuu3]$ | $1 + u$ | 172 |
| \mathcal{K}_3 | $[001u013100301u11u0313003uuu33100]$ | 1 | 174 |
| \mathcal{K}_2 | $[31u30033u03u033u0101u0u11111301u]$ | $1 + u$ | 176 |
| \mathcal{K}_3 | $[3301uu0u30001u1uu3u33313u1031uuu]$ | $1 + u$ | 180 |

TABLE 8. $[68, 34, 12]$ -codes in $W_{68,2}$ by Theorem 4.2 (22 codes)

| X | c | γ | β | X | c | γ | β |
|----------------------|---------|----------|---------|----------------------|---------|----------|---------|
| $[3u3uu3310010u3u0]$ | $1 + u$ | 0 | 160 | $[0u013u3u0000u303]$ | 1 | 0 | 164 |
| $[1uu3313331u001uu]$ | $1 + u$ | 0 | 162 | $[30u33u313uu300u0]$ | $1 + u$ | 0 | 166 |
| $[u13310u0u1100u1u]$ | 1 | 0 | 140 | $[01u33u333u3u3300]$ | 1 | 0 | 168 |
| $[031u133101uu31u0]$ | $1 + u$ | 0 | 144 | $[1010uu30330303u0]$ | $1 + u$ | 1 | 154 |
| $[113010u1u3001130]$ | 1 | 0 | 146 | $[u330uu13uu100uuu]$ | 1 | 1 | 156 |
| $[1u00311u1u131030]$ | $1 + u$ | 0 | 150 | $[uuu01000303uuu11]$ | $1 + u$ | 1 | 158 |
| $[311333300303u13u]$ | 1 | 0 | 152 | $[0u010uu0130u0310]$ | $1 + u$ | 1 | 160 |
| $[0u03000013u03300]$ | 1 | 0 | 154 | $[1330u001uuuu1013]$ | 1 | 1 | 162 |
| $[3u310u1311130u11]$ | $1 + u$ | 0 | 156 | $[uuu10110uu0uu103]$ | 1 | 1 | 164 |
| $[030uu33u1300130u]$ | 1 | 0 | 158 | $[13130u1u0u3uuuu1]$ | 1 | 1 | 170 |
| $[u00uu3u033000103]$ | $1 + u$ | 3 | 176 | $[10u000uu3031003u]$ | 1 | 3 | 196 |

$\mathbb{F}_2 + u\mathbb{F}_2$ -lifts. Let \mathcal{C}_{88} be the binary four circulant code with $r_A = (11101010001)$ and $r_B = (10110101101)$. \mathcal{C}_{88} is lifted to $\mathbb{F}_2 + u\mathbb{F}_2$ and $[88, 44, 16]_2$ extremal doubly even codes with an automorphism group of order 44 are obtained. The codes are different than the previous codes since the order of the automorphism group is different. The inequivalence of the codes is verified by the invariants I_{16} as was done previously. In order to save space we list 10 of the codes in Table 11.

TABLE 9. $[68, 34, 12]$ -codes in $W_{68,2}$ by Theorem 4.1 (13 codes)

| X | c | γ | β |
|--------------------------------------|---------|----------|---------|
| $[30u101113u3131030u10uu0uu0111u33]$ | 1 | 0 | 172 |
| $[u1uu1131u133331330uu01u1uu333100]$ | $1 + u$ | 0 | 176 |
| $[131333uuu0u100u101031u03313uuuu0]$ | 1 | 1 | 148 |
| $[001u0uu131u1uu3u00uuu00u0u003033]$ | $1 + u$ | 1 | 152 |
| $[u30301u01u0u010u311303u3033u0u13]$ | 1 | 1 | 166 |
| $[03333133310u3u0u030013301131011]$ | 1 | 1 | 168 |
| $[0130330100uuu11101u3013uu111u301]$ | $1 + u$ | 1 | 172 |
| $[10uuuu000u30u3u3111u1uu3u3u00030]$ | $1 + u$ | 1 | 174 |
| $[u033030311u0uuu13311uuu030uuuu01]$ | 1 | 1 | 176 |
| $[uuu103031131313100033u01u3010003]$ | 1 | 1 | 178 |
| $[13uu331033u0103uuuu10uu303103133]$ | 1 | 1 | 190 |
| $[0uu0113u00111u00313u3133u1311uuu]$ | $1 + u$ | 1 | 196 |
| $[331u0uu3u10003uuu3u01110u0u31333]$ | 1 | 3 | 188 |

TABLE 10. $[40, 20, 8]_2$ four circulant self-dual codes

| | r_A | r_B | A_8 | I_8 | $ Aut(C) $ |
|-----------------|------------|------------|-------|-------|-------------------|
| \mathcal{D}_1 | 0100011001 | 1110100111 | 285 | 2520 | $2^3 \times 5$ |
| \mathcal{D}_2 | 0100000010 | 1101001001 | 285 | 3090 | $2^3 \times 5$ |
| \mathcal{D}_3 | 0011001011 | 0001101111 | 285 | 2610 | $2^2 \times 5$ |
| \mathcal{D}_4 | 1110001110 | 0100010001 | 285 | 4440 | $2^{14} \times 5$ |
| \mathcal{D}_5 | 1010011111 | 1101011100 | 125 | 360 | $2^3 \times 5$ |
| \mathcal{D}_6 | 0110000110 | 1001001110 | 125 | 390 | $2^3 \times 5$ |
| \mathcal{D}_7 | 1100101010 | 0110100100 | 125 | 570 | $2^2 \times 5$ |

6.3. New doubly even $[96, 48, 16]_2$ codes. A self-dual doubly even $[96, 48, 16]_2$ -code has weight enumerator $1 + (-28086 + \alpha)y^{16} + (3666432 - 16\alpha)y^{20} + \dots$. The first such code with $\alpha = 37722$ is constructed in [7] by a construction from extended binary quadratic residue codes of length 32 and 25 new codes are constructed in [3] via automorphisms of order 23. Let C_{96} be the four circulant code over \mathbb{F}_4 with $r_A = (\omega, 1, 0, 1 + \omega, \omega, \omega)$ and $r_B = (0, 1 + \omega, 1, 0, 1 + \omega, 0)$. C_{96} is a self-dual code which has binary Gray image a $[48, 24, 8]_2$. By lifting this code to $\mathbb{F}_4 + u\mathbb{F}_4$ a family of self-dual codes obtained. As binary Gray images of these codes a substantial number of new doubly-even self dual $[96, 48, 16]_2$ codes are obtained, in order to save space just ten of them are listed in Table 12. The codes in the Tables 11 and 12 are generated by $\left[\begin{array}{c|cc} I_{2n} & A & B \\ \hline & B^T & A^T \end{array} \right]$ over $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$ respectively.

7. CONCLUSION

The binary extension theorems in the literature are used to obtain binary self-dual codes of length $n+2$ from self-dual codes of length n . They have been effectively used to characterize many extremal binary self-dual codes.

In our work, we generalized this extension to rings, since recently self-dual codes over rings have been used to obtain extremal binary self-dual codes. The extension

TABLE 11. New doubly even binary codes of lengths 80 and 88 from $\mathbb{F}_2 + u\mathbb{F}_2$

| \mathcal{L} | \mathcal{C} | r_A | r_B | $ Aut(\mathcal{L}) $ | I_{16} |
|-----------------------|--------------------|---------------|---------------|----------------------|----------|
| $\mathcal{L}_{80,1}$ | \mathcal{D}_1 | [01uuu13uu3] | [111010u131] | $2^4 3 \times 5$ | 20342040 |
| $\mathcal{L}_{80,2}$ | \mathcal{D}_1 | [u3uuu130u3] | [113030u133] | $2^3 5$ | 20062500 |
| $\mathcal{L}_{80,3}$ | \mathcal{D}_1 | [0300033001] | [13301uu313] | $2^3 5$ | 20008440 |
| $\mathcal{L}_{80,4}$ | \mathcal{D}_2 | [01u0uu0u3u] | [1301uu10u3] | $2^3 5$ | 20082720 |
| $\mathcal{L}_{80,5}$ | \mathcal{D}_3 | [0u31u03033] | [uuu13u3113] | $2^3 5$ | 20031600 |
| $\mathcal{L}_{80,6}$ | \mathcal{D}_3 | [0u11003033] | [0uu33u1331] | $2^4 5$ | 20195400 |
| $\mathcal{L}_{80,7}$ | \mathcal{D}_4 | [3330u03130] | [u1u003uu01] | $2^4 5$ | 20207640 |
| $\mathcal{L}_{80,8}$ | \mathcal{D}_4 | [1130u03310] | [u3u0u10u03] | $2^4 5$ | 20306280 |
| $\mathcal{L}_{80,9}$ | \mathcal{D}_5 | [1u3u011131] | [31u1u113u0] | $2^3 5$ | 20003880 |
| $\mathcal{L}_{80,10}$ | \mathcal{D}_5 | [3030013111] | [13u3013100] | $2^4 3 \times 5$ | 20248440 |
| $\mathcal{L}_{80,11}$ | \mathcal{D}_5 | [1u1u011133] | [33030111u0] | $2^4 3 \times 5$ | 20457960 |
| $\mathcal{L}_{80,12}$ | \mathcal{D}_6 | [u1100u031u] | [1u03u03310] | $2^3 5$ | 19992780 |
| $\mathcal{L}_{80,13}$ | \mathcal{D}_6 | [0110uu031u] | [3003003130] | $2^3 5$ | 20021700 |
| $\mathcal{L}_{80,14}$ | \mathcal{D}_7 | [11uu303u10] | [03103u010u] | $2^3 5$ | 20043240 |
| $\mathcal{L}_{88,1}$ | \mathcal{C}_{88} | [13303030003] | [1u31u1033u3] | $2^2 11$ | 1060092 |
| $\mathcal{L}_{88,2}$ | \mathcal{C}_{88} | [13301u30003] | [303103u33u1] | $2^2 11$ | 1078803 |
| $\mathcal{L}_{88,3}$ | \mathcal{C}_{88} | [11303u3u001] | [1u3101u13u1] | $2^2 11$ | 1089990 |
| $\mathcal{L}_{88,4}$ | \mathcal{C}_{88} | [331u1u1u0u3] | [3u31u3u3303] | $2^2 11$ | 1095666 |
| $\mathcal{L}_{88,5}$ | \mathcal{C}_{88} | [13101u3uu01] | [3u33u103103] | $2^2 11$ | 1103553 |
| $\mathcal{L}_{88,6}$ | \mathcal{C}_{88} | [311u3010u03] | [1011u3u31u1] | $2^2 11$ | 1115400 |
| $\mathcal{L}_{88,7}$ | \mathcal{C}_{88} | [311010300u1] | [3u33u1u31u3] | $2^2 11$ | 1132164 |
| $\mathcal{L}_{88,8}$ | \mathcal{C}_{88} | [33103u30001] | [1u11u301303] | $2^2 11$ | 1115664 |
| $\mathcal{L}_{88,9}$ | \mathcal{C}_{88} | [333u1u3uu03] | [101103031u1] | $2^2 11$ | 1128402 |
| $\mathcal{L}_{88,10}$ | \mathcal{C}_{88} | [31103u1uu01] | [3u3101033u3] | $2^2 11$ | 1160181 |

TABLE 12. New $[96, 48, 16]_2$ doubly even codes from $\mathbb{F}_4 + u\mathbb{F}_4$

| $\mathcal{L}_{96,i}$ | r_A | r_B | α |
|-----------------------|----------------------------------|----------------------------------|----------|
| $\mathcal{L}_{96,1}$ | $(b_3, a_1, z_1, c_4, b_4, b_2)$ | $(z_1, c_2, a_2, z_4, c_4, z_1)$ | 36864 |
| $\mathcal{L}_{96,2}$ | $(b_1, a_4, z_1, c_1, b_1, b_1)$ | $(z_2, c_4, a_1, z_2, c_3, z_3)$ | 36876 |
| $\mathcal{L}_{96,3}$ | $(b_2, a_4, z_1, c_3, b_2, b_3)$ | $(z_3, c_4, a_4, z_2, c_2, z_4)$ | 36888 |
| $\mathcal{L}_{96,4}$ | $(b_4, a_2, z_4, c_3, b_3, b_3)$ | $(z_3, c_1, a_3, z_4, c_1, z_3)$ | 36900 |
| $\mathcal{L}_{96,5}$ | $(b_1, a_4, z_2, c_1, b_3, b_3)$ | $(z_3, c_1, a_3, z_2, c_2, z_2)$ | 36912 |
| $\mathcal{L}_{96,6}$ | $(b_4, a_1, z_3, c_2, b_4, b_1)$ | $(z_4, c_2, a_3, z_3, c_4, z_3)$ | 36936 |
| $\mathcal{L}_{96,7}$ | $(b_2, a_1, z_2, c_1, b_2, b_4)$ | $(z_1, c_1, a_2, z_2, c_1, z_3)$ | 36948 |
| $\mathcal{L}_{96,8}$ | $(b_1, a_1, z_1, c_1, b_3, b_1)$ | $(z_2, c_3, a_1, z_3, c_1, z_1)$ | 36960 |
| $\mathcal{L}_{96,9}$ | $(b_2, a_3, z_4, c_2, b_4, b_2)$ | $(z_2, c_2, a_4, z_1, c_1, z_1)$ | 36972 |
| $\mathcal{L}_{96,10}$ | $(b_4, a_1, z_3, c_4, b_3, b_2)$ | $(z_2, c_1, a_4, z_1, c_2, z_2)$ | 36984 |

theorems that we suggest can be applied to all rings of characteristic 2. By using a family of such rings, i.e., $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$, with $m = 1, 2$ and the aforementioned extension theorems we were able to obtain a substantial number of new binary

extremal self-dual codes of certain lengths, the results of which have been tabulated throughout the paper. The results indicate the effectiveness of these extension theorems and thus we believe it will add to the motivation of studying self-dual codes over rings. Working out these extensions in different rings might fill out a lot of the gaps in the study of extremal binary self-dual codes.

A possible line of research could be attempting such extension theorems for rings of other characteristic as well, such as \mathbb{Z}_4 .

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